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On the Temperature Dependence of the Mean Number of Clusters

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The random variable number of cluster defined on the configurations of a ferromagnetic Ising model at zero field and inverse temperature β on a graph $\mathscr{G}$ is considered. The Gibbs average at $\beta = 0$ is proved to be greater than the one at $\beta > 0$ if the degree of $\mathscr{G}$ is not greater than 3.

KEY WORDS: Ferromagnetic Ising model; number of clusters; temperature dependence.

1. DESCRIPTION OF THE RESULT

Let $\mathscr{G} = (V, L)$ be a finite connected graph, where V is the set of the vertices and L the set of the lines. We consider the ferromagnetic Gibbs measure g_β on $\{-1, 1\}^V$ defined by the weight

$$\exp \beta \sum_{\{i,j\} \in L} s(i)\, s(j) \tag{1.1}$$

where $\beta \geqslant 0$ is the inverse temperature. Given $s \in \{-1, 1\}^V$, the $\pm$ clusters of s are the maximal components of $s^{-1}(\pm 1)$ connected in L (including the ones of cardinality 1).

We denote by $C^{\pm}(s)$ the sets of the $\pm$ clusters of s.

Among the events and observables defined essentially in terms of clusters we consider, for instance, the event

$$E_{ij} = \{s \in \{-1, 1\}^V \mid \exists C \in C^+(s) \cup C^-(s) \colon \{i, j\} \subset C\} \tag{1.2}$$

and the random variable "number of clusters"

$$N(s) = |C^+(s)| + |C^-(s)| \tag{1.3}$$

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where $|\cdot|$ denotes cardinality. In this paper we focus on the observable N. For its use in Bernoulli percolation theory we refer to refs. 3 and 4. It is reasonable to conjecture that the g_β probability of cluster events and the mean E_{g_β} with respect to g_β of cluster observables have a monotonic behavior in β, $\beta \geqslant 0$, at least for regular graphs. The setup of the FKG inequalities is not of help in this problem. Actually, the measures g_β are not FKG ordered in β (ref. 5); furthermore, both $N(s)$ and $|C^+(s)|$ are not monotonic functions of s.

On the other hand, the nonlocal character of the cluster observables prevents us from using the Griffiths inequality.[11] One of the applications of the conjectured monotonicity in β of $g_\beta(E_{ij})$ should be the extension of the coexistence in the cubic lattice of the infinite + and − clusters, proved in ref. 1 at small β, up to and above the critical value of β. For the connection with the roughening transition we refer to ref. 2.

The aim of this paper is to show that the mean number of clusters exhibits a monotonicity property in β. We prove the following result.

Proposition 1.1. If $\mathscr{G}$ has in each point degree not greater than 3, then for any $\beta \geqslant 0$,

$$E_{g_\beta}(N) \leqslant E_{g_0}(N) \tag{1.4}$$

We recall that the degree of a point is the number of adjacent points. The most interesting graph to which Proposition 1.1 applies is the hexagonal one.

The basic idea of the proof is that the ferromagnetic Gibbs measure g_β can be represented as a Bernoulli measure conditioned to a group. This ideda was first used in ref. 8. The group we use here is the set of the $\omega \in \{0, 1\}^L$ such that any loop of L contains an even number of 1's of ω. The role of this and related groups in statistical mechanics was recognized in ref. 9 from a point of view in some sense complementary to the present one. The representation of a Gibbs measure as a Bernoulli-conditioned one plays a central role also in the Fortuin–Kastelyn approach to the Ising model (see, for instance, ref. 10), but there the group structure of the conditioning event was not recognized. We mention two results on the monotonicity of cluster events. The ratio of the mean volume to the mean boundary of a cluster is smaller at $\beta = 0$ than for any $\beta > 0$ (ref. 6); the probability of percolation in a lattice gas is monotonic ini β and z (ref. 7) (the range of the parameters is completely different from the present one). Both these results are a straightforward application of the FKG inequality and hence they are in substance quite different from the present one.

2. FROM A GIBBS MEASURE TO A BERNOULLI-CONDITIONED ONE

The space $\{0, 1\}^L$ is a group with respect to the product $\omega_1 \cdot \omega_2$ defined by the rules $(\omega_1 \cdot \omega_2)(l) = \omega_1(l) \cdot \omega_2(l)$ $\forall l \in L$ and $1 \cdot 0 = 0 \cdot 1 = 1$, $0 \cdot 0 = 1 \cdot 1 = 0$. The null configuration is the identity. Define

$$\gamma: \quad s \in \{-1, 1\}^V \to \gamma_s \in \{0, 1\}^L$$

where

$$\gamma_s(\{i, j\}) = \begin{cases} 0 & \text{if} \quad s(i) = s(j) \\ 1 & \text{if} \quad s(i) \neq s(j) \end{cases} \tag{2.1}$$

for any $\{i, j\} \in L$. It is easy to see that $\forall s, t \in \{-1, 1\}^V$

$$\gamma_{st} = \gamma_s \cdot \gamma_t \tag{2.2}$$

where st denotes the ordinary pointwise product of s and t. This implies that the range of γ is a subgroup of $\{0, 1\}^L$, which we denote Γ. This group is characterized by the condition that any loop contains an even number of 1's. Since the weight of any configuration s is proportional to

$$\exp -2\beta |\gamma_s^{-1}(1)| \tag{2.3}$$

the measure g_β can be represented on $\{0, 1\}^L$ as a Bernoulli measure μ_p of parameter $p = (1 + e^{-2\beta})^{-1} e^{-2\beta} \in [0, 1/2]$ conditioned to Γ. With a little abuse of notation we can write

$$g_\beta = \mu_p(\cdot \mid \Gamma) \tag{2.4}$$

If X is a random variable on $\{-1, 1\}^V$, invariant for the total spin flip [i.e., $X(s) = X(-s)$], since

$$\gamma_s = \gamma_t \Rightarrow s = t \quad \text{or} \quad s = -t \tag{2.5}$$

we can define a random variable on Γ, still denoted X, such that $X(s) = X(\gamma_s)$. We thus have

$$E_{g_\beta}(X) = E_{\mu_p}(X \mid \Gamma) \tag{2.6}$$

As the value 0 for the parameter β corresponds to $\frac{1}{2}$ for p, so that

$$E_{g_0}(X) = E_{\mu_{1/2}}(X \mid \Gamma) \tag{2.7}$$

we shall achieve a comparison between $E_{g_\beta}(X)$ and $E_{g_0}(X)$ by means of a comparison between $E_{\mu_p}(X \mid \Gamma)$ and $E_{\mu_{1/2}}(X \mid \Gamma)$, where $p \in [0, 1/2]$.

3. THE EXPECTATION CONDITIONED TO A SUBGROUP

In this section we consider the space $\{0, 1\}^S$, where S is any finite set, with the group structure defined at the beginning of Section 2, and a Bernoulli measure μ_p on it, with $p \in [0, 1/2]$. If $\sigma \neq \varnothing$, $\sigma \subset S$, and $\alpha \in \{0, 1\}^\sigma$, we denote the cylindrical subsets of $\{0, 1\}^S$ by

$$\binom{\sigma}{\alpha} = \{\omega \in \{0, 1\}^S \mid \omega(i) = \alpha(i) \quad \forall i \in \sigma\} \tag{3.1}$$

We also include in this notation the case $\sigma = \varnothing$, defining $\{0\} = \{0, 1\}^\varnothing$ and $\binom{\varnothing}{0} = \{0, 1\}^S$. If X is a random variable on $\{0, 1\}^S$ and G is a subgroup, we define the random variable X_G on the subsets of S putting

$$X_G(\sigma) = \left| G \cap \binom{\sigma}{0} \right|^{-1} \sum_{\omega \in G \cap \binom{\sigma}{0}} X(\omega) \tag{3.2}$$

Since G is a group, $G \cap \binom{\sigma}{0}$ is also a group and, in particular, it is nonempty. Furthermore, one has

$$X_G(\varnothing) = |G|^{-1} \sum_{\omega \in G} X(\omega) \tag{3.3}$$

Lemma 3.1. If G is a group and $p \in [0, 1/2]$, a sufficient condition for

$$E_{\mu_p}(X \mid G) \geqslant E_{\mu_{1/2}}(X \mid G)$$

is, $\forall \sigma \subset S$,

$$X_G(\sigma) \geqslant X_G(\varnothing) \tag{3.4}$$

Proof. Denoting for brevity $\omega = \omega^{-1}(1)$ and $\omega^c = \omega^{-1}(0)$, we have

$$\begin{aligned} \mu_p(\omega) &= p^{|\omega|}(1 - 2p + p)^{|\omega^c|} \\ &= \sum_{\sigma \subset \omega^c} (1 - 2p)^{|\sigma|} p^{|S \backslash \sigma|} \end{aligned}$$

Hence

$$\sum_{\omega \in G} \mu_p(\omega) X(\omega) = \sum_{\sigma \subset S} (1 - 2p)^{|\sigma|} p^{|S \backslash \sigma|} \sum_{\omega \in G \cap \binom{\sigma}{0}} X(\omega)$$

From this equality, if we define, $\forall \sigma \subset S$,

$$v_p^G(\sigma) = \mu_p(G)^{-1}(1 - 2p)^{|\sigma|} p^{|S \backslash \sigma|} \left| G \cap \binom{\sigma}{0} \right| \tag{3.5}$$

we get

$$E_{\mu_p}(X \mid G) = \sum_{\sigma \subset S} X_G(\sigma)\, \nu_p^G(\sigma) \tag{3.6}$$

The measure ν_p^G is a probability measure on the subsets of S, as can be easily checked putting $X = 1$ in Eq. (3.6). By the definition it is also immediate that $\nu_{1/2}^G$ is the delta measure concentrated on $\sigma = \varnothing$. The ν_p^G has the nice property of being FKG,[8] but we do not use it here. From (3.6) we thus get

$$E_{\mu_{1/2}}(X \mid G) = X_G(\varnothing) \tag{3.7}$$

The lemma follows from Eqs. (3.6) and (3.7).

We remark that the hypothesis that G is a group has been only used in $G \cap \binom{\sigma}{0} \neq \varnothing$. In the sequel we shall use also the following property: given $\alpha \in \{0, 1\}^{\sigma}$, if $G \cap \binom{\sigma}{\alpha}$ is nonempty, it is a coset of $G \cap \binom{\sigma}{0}$, and so

$$\left| G \cap \begin{pmatrix} \sigma \\ \alpha \end{pmatrix} \right| = \left| G \cap \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \right| \tag{3.8}$$

If $l \in \sigma$, we denote for brevity by $\sigma \backslash l$ and l, respectively, the sets $\sigma \backslash \{l\}$ and $\{l\}$.

Lemma 3.2. In order to have inequality (3.4), it is sufficient that $\forall \sigma \subset S$, $\sigma \neq \varnothing$, $\exists l \in \sigma$ such that

$$G \cap \begin{pmatrix} \sigma \backslash l & l \\ 0 & 1 \end{pmatrix} = \varnothing$$

or

$$\sum_{\omega \in G \cap \binom{\sigma \backslash l \;\; l}{0 \;\;\; 0}} X(\omega) \geqslant \sum_{\omega \in G \cap \binom{\sigma \backslash l \;\; l}{0 \;\;\; 1}} X(\omega) \tag{3.9}$$

Proof. Inequality (3.4) is implied by the following, $\forall \sigma \subset S$, $\sigma \neq \varnothing$, $\exists l \in \sigma$:

$$X_G(\sigma) \geqslant X_G(\sigma \backslash l) \tag{3.10}$$

which, by the definition of X_G, is equivalent to

$$\left| G \cap \begin{pmatrix} \sigma \backslash l \\ 0 \end{pmatrix} \right| \sum_{\omega \in G \cap \binom{\sigma}{0}} X(\omega) \geqslant \left| G \cap \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \right| \sum_{\omega \in G \cap \binom{\sigma \backslash l}{0}} X(\omega) \tag{3.11}$$

We apply to (3.11) the following equations:

$$\left|G\cap\begin{pmatrix}\sigma\backslash l\\0\end{pmatrix}\right|=\left|G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}\right|+\left|G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 1\end{pmatrix}\right|$$
$$\sum_{\omega\in G\cap\begin{pmatrix}\sigma\backslash l\\0\end{pmatrix}}X(\omega)=\sum_{\omega\in G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}}X(\omega)+\sum_{\omega\in G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 1\end{pmatrix}}X(\omega) \tag{3.12}$$

with the obvious remark that

$$\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}=\begin{pmatrix}\sigma\\0\end{pmatrix}$$

We thus get that inequality (3.11) is equivalent to

$$\left|G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 1\end{pmatrix}\right|\sum_{\omega\in G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}}X(\omega)\geqslant\left|G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}\right|\sum_{\omega\in G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 1\end{pmatrix}}X(\omega) \tag{3.13}$$

We conclude the proof by using the above-stated property of

$$G\cap\begin{pmatrix}\sigma\backslash l & l\\1 & 1\end{pmatrix}$$

If this set is empty, both members of inequality (3.13) are zero; if it is nonempty, it is a coset of

$$G\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}$$

and they have the same cardinality.

4. PROOF OF PROPOSITION 1.1

The results of the previous section, when applied to our case, can be summarized as follows. A sufficient condition to have

$$E_{g\beta}(-N)\geqslant E_{g0}(-N) \tag{4.1}$$

is that $\forall\sigma\subset L$, $\sigma\neq\varnothing$, $\exists l\in\sigma$:

$$\Gamma\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 1\end{pmatrix}=\varnothing$$

or

$$\sum_{\omega\in\Gamma\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 0\end{pmatrix}}-N(\omega)\geqslant\sum_{\omega\in\Gamma\cap\begin{pmatrix}\sigma\backslash l & l\\0 & 1\end{pmatrix}}-N(\omega) \tag{4.2}$$

By using the characterization of Γ given in Section 2, we get that if σ contains a loop, say τ, $\forall l \in \tau$ one has

$$\Gamma \cap \begin{pmatrix} \sigma \backslash l & l \\ 0 & 1 \end{pmatrix} = \varnothing$$

Therefore we can consider only those σ's that do not contain any loop, i.e., the connected components of σ are trees. Let $\tau_l \subset \sigma$ be the tree to which l belongs. We choose l to be an endline of τ_l. On the other hand, we can rewrite inequality (4.2) in the spin language:

$$\sum_{s \in \begin{bmatrix} \sigma \backslash l & l \\ = & = \end{bmatrix}} N(s) \leqslant \sum_{s \in \begin{bmatrix} \sigma \backslash l & l \\ = & \neq \end{bmatrix}} N(s) \tag{4.3}$$

where we have defined, for instance,

$$\begin{bmatrix} \sigma \backslash l & l \\ = & \neq \end{bmatrix} = \{ s \in \{-1, 1\}^V \mid \{i, j\} \in \sigma \backslash l \Rightarrow s(i) = s(j); \{i, j\} = l \Rightarrow s(i) \neq s(j) \}$$

and we have dropped a factor 2 from both the members. If $l = \{i, j\}$, denoting $\alpha = (\bigcup_{b \in \tau_l} b) \backslash j$ (i.e., α is the set of the vertices of τ_l minus j), where j is an endpoint of τ_l, and denoting for brevity

$$\Sigma_{V \backslash (\alpha \cup j)} = \{-1, 1\}^{V \backslash (\alpha \cup j)} \cap \begin{bmatrix} \sigma \backslash \tau_l \\ = \end{bmatrix}$$

inequality (4.3) is equivalent to

$$\sum_{s \in \Sigma_{V \backslash (\alpha \cup j)}} [N(s^{\alpha\ j}_{+\ +}) + N(s^{\alpha\ j}_{-\ -})] \leqslant \sum_{s \in \Sigma_{V \backslash (\alpha \cup j)}} [N(s^{\beta\ j}_{+\ -}) + N(s^{\alpha\ j}_{-\ +})] \tag{4.4}$$

where we have defined, for instance, $s^{\alpha\ j}_{+\ -}$ the configuration of $\{-1, 1\}^V$ obtained completing $s \in \{-1, 1\}^{V \backslash (\alpha \cup j)}$ with $+1$ in α and -1 in j. A sufficient condition to have inequality (4.4) is, $\forall s \in \{-1, 1\}^{V \backslash (\alpha \cup j)}$,

$$N(s^{\alpha\ j}_{+\ +}) + N(s^{\alpha\ j}_{-\ -}) \leqslant N(s^{\alpha\ j}_{+\ -}) + N(s^{\alpha\ j}_{-\ +}) \tag{4.5}$$

If α and δ are connected adjacent subsets of V, we consider the inequality

$$N(s^{\alpha\ \delta}_{+\ +}) + N(s^{\alpha\ \delta}_{-\ -}) \leqslant N(s^{\alpha\ \delta}_{+\ -}) + N(s^{\alpha\ \delta}_{-\ +}) \tag{4.6}$$

that generalizes the previous one. Given $s \in \{-1, 1\}^{V \backslash (\alpha \cup \delta)}$, we denote by $C^{\pm}_{\alpha}(s)$ the set of clusters of $s^{-1}(\pm 1)$ adjacent to α and by $\bar{N}(s)$ the number of clusters of s. We have

$$\begin{aligned} N(s^{\alpha\ \delta}_{+\ +}) &= \bar{N}(s) - |C^+_{\alpha}(s) \cup C^+_{\delta}(s)| + 1 \\ N(s^{\alpha\ \delta}_{+\ -}) &= \bar{N}(s) - |C^+_{\alpha}(s)| + 1 - |C^-_{\delta}(s)| + 1 \end{aligned} \tag{4.7}$$

Using these equations and the obvious one

$$|C_\alpha^+(s) \cup C_\delta^+(s)| = |C_\alpha^+(s)| + |C_\delta^+(s)| - |C_\alpha^+(s) \cap C_\delta^+(s)|$$

we get the following inequality equivalent to (4.6):

$$|C_\alpha^+(s) \cap C_\delta^+(s)| + |C_\alpha^-(s) \cap C_\delta^-(s)| \leqslant 2 \tag{4.8}$$

In the particular case $\delta = \{j\}$, inequality (4.5) is thus equivalent to, $\forall s \in \{-1, 1\}^{V\setminus(\alpha \cup j)}$,

$$|C_\alpha^+(s) \cap C_j^+(s)| + |C_\alpha^-(s) \cap C_j^-(s)| \leqslant 2 \tag{4.9}$$

The left-hand side of inequality (4.9) is bounded, uniformly in s, by the number of points of $V\setminus(\alpha \cup j)$ that are simultaneously adjacent to α and j. This number is not greater than 2, since by hypothesis the degree of j is not greater than 3, and α and j have been chosen to be adjacent. Hence inequality (4.9) is true and this concludes the proof.

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